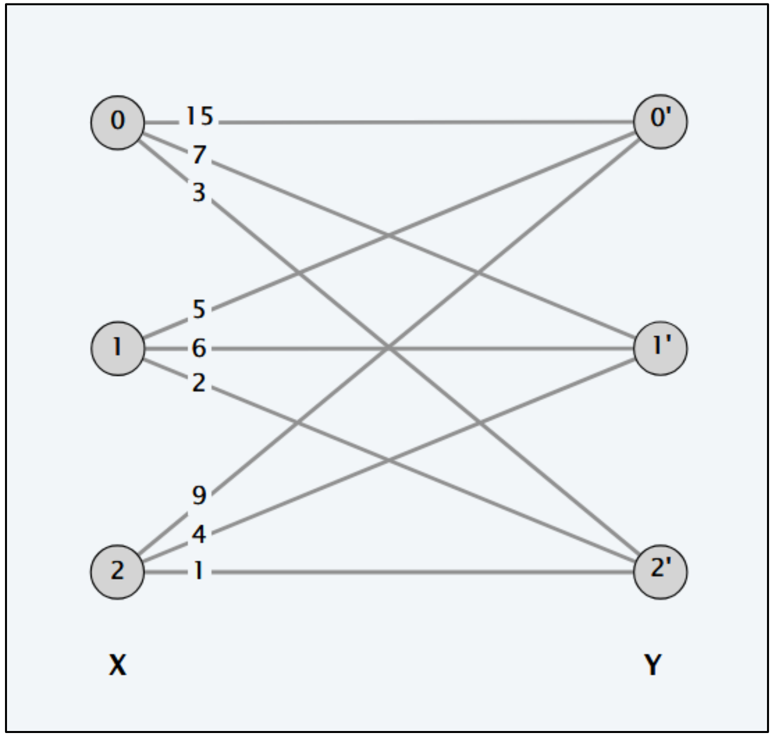
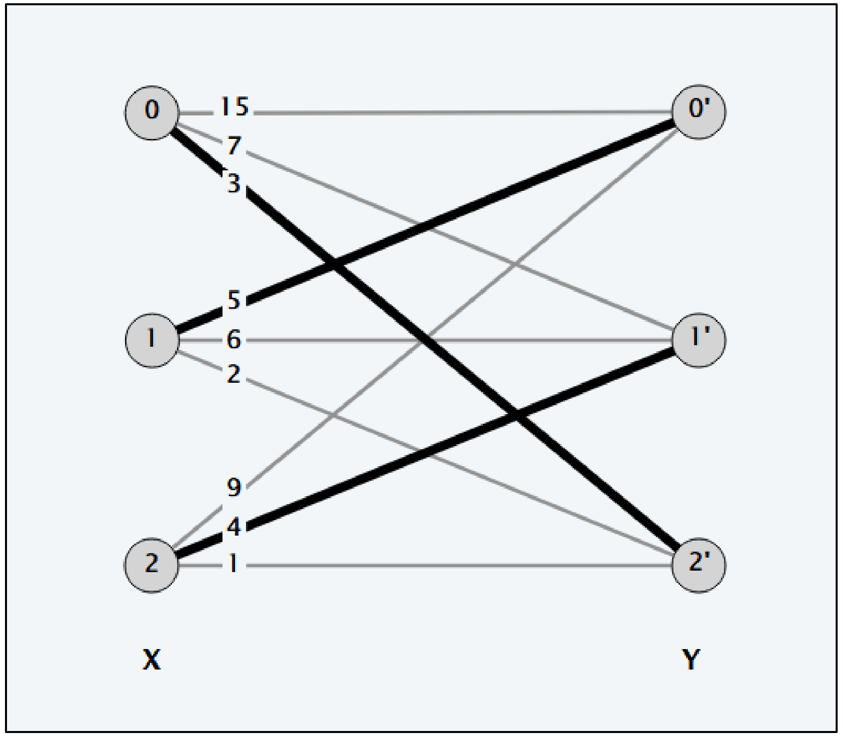
Minimum Cost Bipartite Perfect Matching

AKA “the assignment problem”





Monge [1784] wanted an optimal way to transport earth

Lorsqu'on doit transporter des terres d'un lieu dans un autre, on a coutime de donner le nom de *Déblai* au volume des terres que l'on doit transporter, & le nom de *Remblai* à l'espace qu'elles doivent occuper après le transport.

When one must transport earth from one place to another, one usually gives the name of *Déblai* to the volume of earth that one must transport, & the name of *Remblai* to the space that they should occupy after the transport.

The price of the transport of one molecule being, if all the rest is equal, proportional to its weight & to the distance that one makes it covering, & hence the price of the total transport having to be proportional to the sum of the products of the molecules each multiplied by the distance covered, it follows that, the déblai & the remblai being given by figure and position, it makes difference if a certain molecule of the déblai is transported to one or to another place of the remblai, but that there is a certain distribution to make of the molecules from the first to the second, after which the sum of these products will be as little as possible, & the price of the total transport will be a *minimum.*

First known correct algorithm published due to Easterfield [1946].

Assigning units to postings in WW2:

> In the course of a piece of organisational research into the problems of demobilisation in the R.A.F., it seemed that it might be possible to arrange the posting of men from disbanded units into other units in such a way that they would not need to be posted again before they were demobilised; and that a study of the numbers of men in the various release groups in each unit might enable this process to be carried out with a minimum number of postings. Unfortunately the unexpected ending of the Japanese war prevented the implications of this approach from being worked out in time for effective use. The algorithm of this paper arose directly in the course of the investigation.

Running time $O(n^2 \cdot 2^n)$

Birkhoff [1946] was studying doubly stochastic matrices

Relevant for Random walks / Markov chain algorithms:

Birkhoff [1946] derived from Hall's marriage theorem that each doubly stochastic matrix is a convex combination of permutation matrices. Birkhoff's motivation was:

> Estas matrices son interesantes para la probabilidad, y los cuadrados mágicos son múltiplos escalares de estas matrices.[33]

[33] These matrices are interesting because of the probability, and the magic squares are scalar multiples of these matrices.

Birkhoff, Garrett (1946), "Tres observaciones sobre el algebra lineal, Univ. Nac. Tucumán.

- Speech at the American Psychological Association, Denver, Colorado
- Thorndike [1950] studied the problem of "classification' of personnel":

> The past decade, and particularly the war years, have witnessed a great concern about the classification of personnel and a vast expenditure of effort presumably directed towards this end.

> There are, as has been indicated, a finite number of permutations in the assignment of men to jobs. When the classification problem as formulated above was presented to a mathematician, he pointed to this fact and said that from the point of view of the mathematician there was no problem. Since the number of permutations was finite, one had only to try them all and choose the best. He dismissed the problem at that point. This is rather cold comfort to the psychologist, however, when one considers that only ten men and ten jobs mean over three and a half million permutations. Trying out all the permutations may be a mathematical solution to the problem, it is not a practical solution.

The Hungarian Algorithm

Harold Kuhn [1956] achieved a "good algorithm" for the assignment problem.

Based on the work of Egervary, Kuhn called it the "Hungarian method".

Kuhn:

> During this period, I was reading Kőnig's classical book on the theory of graphs and realized that the matching problem for a bipartite graph on two sets of n vertices was exactly the same as an n by n assignment problem

- Kuhn, continued:

Reading Kőnig's book more carefully, I was struck by the following footnote (p. 238, footnote 2): "... Eine Verallgemeinerung dieser Sätze gab Egerváry, Matrixok kombinatorius tulajdonságairól |(Über kombinatorische Eigenschaften von Matrizen), Matematikai és Fizikai Lapok, *38*, 1931, S. 16-28 (ungarisch mit einem deutschen Auszug) ..." This indicated that the key to the problem might be in Egerváry's paper. When I returned to Bryn Mawr College in the fall, I obtained a copy of the paper together with a large Hungarian dictionary and grammar from the Haverford College library. I then spent two weeks learning Hungarian and translated the paper [1]. As I had suspected, the paper contained a method by which a general assignment problem could be reduced to a finite number of 0-1 assignment problems.

Kuhn:

> Using Egerváry's reduction and Kőnig's maximum matching algorithm, in the fall of 1953 I solved several 12 by 12 assignment problems (with 3-digit integers as data) by hand. Each of these examples took under two hours to solve and I was convinced that the combined algorithm was 'good'. This must have been one of the last times when pencil and paper could beat the largest and fastest electronic computer in the world.